

MAS212 Linear Algebra I 2007-2008

Coursework 4

Please hand in your solutions of the **starred** exercises by **2pm on Friday 26 October** using the [blue Linear Algebra I Collection Box](#) on the Ground Floor.

Don't forget to put your **name** and **student number** on your solutions, and to **staple** them.

Exercise* 1. (7 marks) Let A and B be 2×2 matrices. Show directly from the definition of the determinant for 2×2 matrices that the following three statements hold:

- (a) if B is obtained from A by swapping row 1 and row 2, then $\det(B) = -\det(A)$;
- (b) if B is obtained from A by multiplying row 1 by α , then $\det(B) = \alpha \det(A)$;
- (c) if B is obtained from A by adding a multiple of row 2 to row 1, then $\det(B) = \det(A)$.

Explain how to use the above results to deduce the theorem on the behaviour of determinants under row operations (Theorem 3.11 in the notes) for 2×2 matrices.

Exercise* 2. (2 marks)

- (a) Let A be an invertible matrix. Show that $\det(A^{-1}) = \frac{1}{\det(A)}$.
- (b) Let A be a square matrix. Show that if $B = S^{-1}AS$ for some invertible matrix S , then $\det(A) = \det(B)$.

Exercise* 3. (4 marks) Use Cramer's rule to compute the solution set of the following linear system:

$$\begin{array}{rclcl} 2x_1 & + & x_2 & & = 7 \\ -3x_1 & & & + & x_3 = -8 \\ & & x_2 & + & 2x_3 = -3 \end{array}$$

Exercise* 4. (3 marks for (a) and (b), none for (c)) Let

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & 2 \end{pmatrix}.$$

- (a) Calculate $\det(A)$ and deduce that A is invertible.
- (b) Determine the $(1,3)$ -entry of A^{-1} by computing a quotient of two determinants.
- (c) Compute the adjugate of A and, hence, determine A^{-1} .

Exercise* 5. (4 marks) Let $C^1[a, b]$ denote the subset of those functions in $C[a, b]$ that have a continuous derivative on $[a, b]$. If addition and scalar multiplication are defined as in $C[a, b]$ then $C^1[a, b]$ turns out to be a vector space. Explain why (C1) and (C2) are satisfied. What is the zero element in $C^1[a, b]$? Given $\mathbf{f} \in C^1[a, b]$ what is $-\mathbf{f}$? Show that (A4) is satisfied.

Exercise 6. Let S be the set of all ordered pairs of real numbers and define the following two operations on S

$$\begin{aligned} \alpha(x_1, x_2) &= (\alpha x_1, \alpha x_2) \quad (\text{'scalar multiplication'}) \\ (x_1, x_2) \oplus (y_1, y_2) &= (x_1 + y_1, 0) \quad (\text{'addition'}) \end{aligned}$$

Show that S equipped with the above operations is not a vector space.

[Prize question overleaf]

A cash prize is offered for the best solution received to the following question. Do not hand it in with your coursework, but hand it to me in person either at the Friday lecture or before.

Prize question. If A_{11} , A_{12} , A_{21} , and A_{22} are $n \times n$ matrices, we can form a $2n \times 2n$ matrix $M = (m_{ij})$, called a **partitioned** (or **block**) **matrix**

$$M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$\begin{aligned} m_{11} &= \text{the } (1, 1)\text{-entry of } A_{11} \\ m_{1,n+1} &= \text{the } (1, 1)\text{-entry of } A_{12} \\ m_{n+1,1} &= \text{the } (1, 1)\text{-entry of } A_{21} \\ m_{n+1,n+1} &= \text{the } (1, 1)\text{-entry of } A_{22} \end{aligned}$$

and so forth. Partitioned matrices play an important role in modern applied linear algebra. Convince yourself that partitioned matrices can be added and multiplied just as you expect. More precisely, if $B_{ij} \in \mathbb{R}^{n \times n}$ for $1 \leq i, j \leq 2$, then

$$\begin{aligned} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} &= \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix} \\ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} &= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}. \end{aligned} \quad (1)$$

Note, however, that in (1) the order of the factors plays a role, since the A_{ij} 's and B_{ij} 's need not commute.

Suppose now that A , B , C and D are arbitrary $n \times n$ matrices. Show that

$$\det \begin{pmatrix} A & B \\ O & D \end{pmatrix} = \det(AD),$$

and find a similar formula (careful!) for

$$\det \begin{pmatrix} O & B \\ C & D \end{pmatrix}.$$

Finally, show that

$$\det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det((A+B)(A-B)).$$

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